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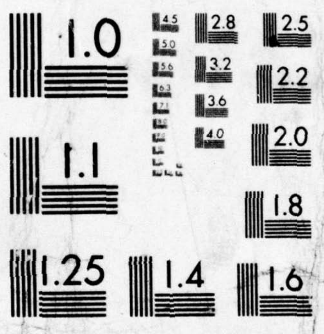

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A method for calculating the electromagnetic field scattered from a rough, perfectly conducting surface having two-dimensional variation is presented. The calculation proceeds in two steps: (1) calculation of the current distribution on the surface via integral equation techniques; and (2) calculation of the scattered fields using these currents as sources. Shadowing and multiple scattering effects are included.

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Final Technical Report

REFLECTION OF ELECTROMAGNETIC RADIATION FROM THE OCEAN SURFACE

ONR Contract No. N00014-73-C-0304

NR294-018

by

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30 October 1976

1. INTRODUCTION

The purpose of this report is to provide a method for calculating the scattered electromagnetic field produced by illumination of a rough surface. At the outset it will be appropriate to list the restrictions imposed on the problem.

As to the incident field, the theory is formulated for a general form of illumination. There is no restriction in incidence angle or on type of polarization. Over a substantial portion of the report, mostly for purposes of comparison with other work, the incident field will be taken to be a plane wave with simple polarization (usually TM). When periodicity is assumed for the surface, the incidence angle of the illuminating radiation will be restricted to certain angles in order to achieve an overall periodicity compatible with the surface.

The restrictions imposed on the surface are that it is assumed to be of infinite extent, i.e., when given explicitly by $z = f(x,y)$ we allow $-\infty \leq x \leq \infty$ and $-\infty \leq y \leq \infty$. The surface is further assumed to be the interface between vacuum ($z > f(x,y)$, index of refraction equal to one) and a perfect conductor ($z < f(x,y)$ conductivity infinite). Finally, we assume that $f(x,y)$ and its first and second derivatives are continuous. Where necessary for comparison with other work, it may be assumed that the surface is representable as a Fourier integral with the attendant requirement that $|f(x,y)| \to 0$ as $|x + y| \to \infty$. When periodicity in $f(x,y)$ is assumed the Fourier integral becomes the usual Fourier sum. Note that no assumption is made concerning the smallness of the surface variation on slopes. The only surfaces excluded are those having sharp points or ridges where surface curvature becomes infinite (local radius of curvature equal zero).

Formulae are given for the calculation of the scattered field arbitrarily close to the surface. Indeed, without this it is not possible to discuss the full spectrum of the scattered field in the "valleys" of the surface,

i.e., in the region where z lies between maximum and minimum $f(x,y)$, ($0 = f_{min} < z < f_{max}$). Upon taking the asymptotic form of certain Green's functions appearing in the expression for the scattered field, the somewhat simpler forms of the scattered field remote from the surface are easily extracted.

The general method used in this work is the exact theory for the reflection problem provided by a two stage calculation. First, one solves the appropriate integral equation system for the surface currents induced by the incident field. Second, these currents used as sources furnish the structure of the scattered field by means of the usual Green's function techniques. In this way it is possible to analyze the conventional, or physical optics, approach to the scattering problem and determine the range of its validity. It will be found that even apart from the restriction of small surface roughness, there is a basic assumption of the physical optics approach which is invalid. In order to examine this point further we turn to a very brief recapitulation of the physical optics approach in simple enough form to exhibit the difficulties.

2. THE PHYSICAL OPTICS SOLUTION

The method presented in this section is perhaps best exemplified by the work of Rice[1] although the basic idea is much older; the same approach, used for acoustical scattering, goes back at least to Lord Rayleigh.[2] Variants on Rice's original presentation have been attempted in an effort to relax some of the basic conditions. For example, Barrick[3] has applied Leontovich[4] boundary conditions in order to move away from the restrictions of perfect conductivity of the surface. It may be noted here that the literature on this problem is very extensive and that no review of the literature will be undertaken in this report.

In our example we consider the surface given by $z = f(x)$ with the normal,

$$\vec{n} = \vec{n}(x) = (1 + f_x^2)^{-\frac{1}{2}} [-f_x \vec{\varepsilon}_x + \vec{\varepsilon}_z] \tag{1}$$

$f_x = df/dx$; ε_x, ε_y, and ε_z are unit vectors in the x, y, and z directions respectively. For the TM mode ($E_y = 0$) the boundary condition of no tangential component of electric field on the surface assumes the simple form,

$$n_z E_x - n_x E_z = 0 \qquad \text{on } z = f(x) \tag{2}$$

which, upon retention of first order terms assuming f_x is small, becomes

$$E_x + f_x E_z = 0 \qquad\qquad \text{on } z = f(x) \tag{3}$$

The <u>incident field</u> has only a y component,

$$\vec{H}^{inc} = H_o \vec{\varepsilon}_y e^{ik[x \cos \theta_o - z \sin \theta_o]} = H_o \vec{\varepsilon}_y \Phi^{inc} \tag{4}$$

It may be noted that θ_o is measured from the horizontal. The corresponding electric field is easily found:

$$E_x^{inc} = - H_o \sin \theta_o \, e^{ik\,[x \cos \theta_o - z \sin \theta_o]} = - \frac{i}{k} H_o \frac{\partial \Phi^{inc}}{\partial z}$$

$$E_z^{inc} = - H_o \cos \theta_o \, e^{ik\,[x \cos \theta_o - z \sin \theta_o]} = \frac{i}{k} H_o \frac{\partial \Phi^{inc}}{\partial x} \tag{5}$$

A time dependence factor, e^{-ikct}, has been assumed for all field variables where $k = 2\pi/\lambda$ and λ is the wavelength of the incident radiation.

For the scattered field the physical optics approach assumes a superposition of upward propagating waves, each of which is in form similar to that suggested by eq. (4). The function corresponsing to Φ^{inc} is

$$\Phi^s(x,z) = \int_{-\infty}^{\infty} d\rho\, A(\rho)\, e^{ik[x\rho + z\sqrt{1-\rho^2}]} \qquad (z > 0) \qquad (6)$$

For $0 < |\rho| < 1$, these are waves scattered upward at some fixed angle with amplitude proportional to $A(\rho)$. For $|\rho| > 1$, the scattered field consists of trapped waves on the surface decaying in amplitude exponentially with increasing positive z.

The procedure is now straight forward. The scattered electric field components, E_x^s and E_z^s, are calculated from equations similar to eqs. (5) using Φ^s in place of Φ^{inc}. Then forming the total field

$$E_x^{tot} = E_x^{inc} + E_x^s \; ; \quad E_z^{tot} = E_z^{inc} + E_z^s$$

the boundary conditions, eq. (3) are applied. Since $f(x)$ as well as its first derivative is assumed small, an expansion in powers of f and f_x is made, and a corresponding perturbation expansion of $A(\rho)$ is made, i.e., $A(\rho) = A_o(\rho) + A_1(\rho) + \dots$ $A_o(\rho)$ represents the contribution of a flat surface ($f(x) \equiv 0$) while $A_1(\rho)$ is the contribution from first order terms in $f(x)$ and $f_x(x)$.

The zeroth order yields simply,

$$A_o(\rho) = \delta(\rho - \cos\theta_o) \qquad (7)$$

where $\delta(x)$ is the usual Dirac delta function. The corresponding equation for

first order term is,

$$\int_{-\infty}^{\infty} d\rho \sqrt{1 - \rho^2}\, A_1(\rho)\, e^{ikx\rho} = - 2[ikf \sin^2 \theta_o - f_x \cos \theta_o]\, e^{ikx \cos \theta_o} \tag{8}$$

so that an explicit solution for $A_1(\rho)$ follows from the Fourier expansion of $f(x)$. Suppose $f(x)$ has a horizontal scale Λ and vertical scale σ so that,

$$f(x) = \sigma \int_{-\infty}^{\infty} d\kappa\, c(\kappa)\, e^{i\kappa(x/\Lambda)} \tag{9}$$

then with the definitions $\alpha = k\Lambda$, $\beta = k\Lambda$, and $\varepsilon = \sigma/\Lambda$ eq. (7) yields,

$$A_1(\rho) = \varepsilon\, c(\alpha(\rho - \cos \theta_o))\, [2i\alpha^2(\rho \cos \theta_o - 1)]\, (1 - \rho^2)^{-\frac{1}{2}} \tag{10}$$

The smallness of $A_1(\rho)$ in comparison with $A_o(\rho)$ is indicated by the factor ε in eq. (10). When restriction to one-dimensional surface variation is made, the above result is that of Rice. Extension to second order is straightforward and will not be repeated here.

The important point to note is the unwarranted nature of the assumption of upward propagating waves only (eq. (6)) coupled with application of the boundary condition in a region ($f_{min} \leqslant z \leqslant f_{max}$) where the whole spectrum cannot be represented by such a hypothesis. The physical optics approach provides no method by which the requisite downward propagating waves can be introduced. It is one of the purposes of this report to show explicitly how these downward propagating components make their appearance. The successive order perturbation terms of the physical optics approach can be interpreted as a form of multiple scattering contributions, but the interpretation is suggestive more than real

since the successive terms correspond simply to iteration of a linearized (in ε) integral equation.

3. THE INTEGRAL EQUATIONS FOR SURFACE MAGNETIC FIELDS

Rather than make any assumptions about the nature of the scattered field, and consequently the total field on the surface, the viewpoint taken here emphasizes the direct calculation of these quantities. The basic formula for this procedure is the Chu-Stratton integral equation for the surface magnetic field. The second stage of the calculation uses these surface fields as sources for the application of a suitable Green's function technique to yield the scattered field. The integral equation for a perfect conductor is

$$\vec{H}(x') = 2\vec{H}^{ext}(x') + \frac{1}{2\pi} k^3 \int d\sigma \, [\vec{n}(x) \times \vec{H}(x)] \times \vec{r}(x,x') \, \frac{ikr - 1}{(kr)^3} e^{ikr} \tag{11}$$

where $\vec{x}$ symbolizes the vector with components $(x,y,f(x,y))$ and similarly for $\vec{x}'$, $\vec{r}(x, x') = \vec{x} - \vec{x}'$, and the integration is over the whole surface. The vector $\vec{n}\, d\sigma$ has components $(-f_x, -f_y, 1)\, dx\, dy$. From the surface boundary condition, already used in eq. (11), $\vec{n} \cdot \vec{H} = 0$ it is evident that the third component of the magnetic field is determined in terms of H_x and H_y so that only two coupled integral equations for H_x and H_y suffice to give the whole surface field.

$\vec{H}^{ext}(x)$ is the incident magnetic field which, for plane wave illumination, has x and y components.

$$(H_x^{ext}, H_y^{ext}) = H_o(\sin\theta_o, -\Gamma_o) \exp\left[ik(x\cos\theta_o - f(x,y)\sin\theta_o\right] \tag{12}$$

with Γ_o a complex number dependent on polarization conditions; H_o is the scale

of the incident field. Some possibilities are: $\Gamma_o = 0$ gives TE mode, $\Gamma_o = \pm i$ gives right (left) circular polarization, while the replacement of $(\sin\theta_o, -\Gamma_o)$ by $(0, -1)$ gives the TM mode.

The use of eq. (11) is much simplified by the introduction of scaled variables. Let Λ and σ be horizontal and vertical scales respectively of $f(x,y)$. So let $x = \Lambda\xi$ and $y = \Lambda\eta$, and set $f(x,y) = \sigma\tau(\xi,\eta)$. The derivatives of $f(x,y)$ scale according to $f_x(x,y) = \varepsilon\tau_\xi(\xi,\eta)$; $f_y(x,y) = \varepsilon\tau_\eta(\xi,\eta)$; $f_{xx}(x,y) = \varepsilon^2\tau_{\xi\xi}(\xi,\eta)$, etc., where $\varepsilon = \beta/\alpha$. The same notation as follows eq. (9) in relating these scale factors to incident wavelength is also used so $\alpha = k\Lambda$, and $\beta = k\sigma$. The magnetic fields are scaled by H_o so on the surface,

$$(H_x(x,y,\, f(x,y)),\quad H_y(x,y,\, f(x,y)) = H_o(\psi_1(\xi,\eta),\, \psi_2(\xi,\eta))$$

$$(H_x^{ext}(x,y,f(x,y)),\quad H_y^{ext}(x,y,\, f(x,y)) = H_o(\sin\theta_o,\, -\Gamma_o)\, e^{i\alpha[\xi\cos\theta_o - \varepsilon\tau(\xi,\eta)\sin\theta_o]} \tag{13}$$

The expression $[(ikr - 1)\exp ikr]/(kr)^3$ is replaced by $(-i\sqrt{2/\pi})H_{3/2}^{(1)}(\alpha\rho_o)/(\alpha\rho_o)^{3/2}$ where, with $\tau' = \tau(\xi',\eta')$,

$$\rho_o = [(\xi-\xi')^2 + (\eta-\eta')^2 + \varepsilon^2(\tau-\tau')^2]^{\frac{1}{2}} \tag{14}$$

and $H_{3/2}^{(1)}$ is the Hankel function of the first kind of order 3/2.

Introducing the column vector $\psi(\xi,\eta)$ whose components are $\psi_1(\xi,\eta)$ and $\psi_2(\xi,\eta)$ allows eq. (11) to be written

$$\psi(\xi',\eta') = 2\psi^{ext}(\xi',\eta') - i\,\frac{1}{2\sqrt{2\pi}}\,\varepsilon\alpha^3 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \frac{H_{3/2}^{(1)}(\alpha\rho_o)}{(\alpha\rho_o)^{3/2}}\, M(\xi',\eta';\xi,\eta)\,\psi(\xi,\eta) \tag{15}$$

A similar column vector has been introduced for the incident field

$$\psi^{ext}(\xi',\eta') = \begin{Vmatrix} \sin\theta_o \\ -\Gamma_o \end{Vmatrix} e^{i\alpha[\xi\cos\theta_o - \varepsilon\tau(\xi,\eta)\sin\theta_o]}$$

$$= \psi_o^{ext}\, e^{i\alpha[\xi\cos\theta_o - \varepsilon\tau(\xi,\eta)\sin\theta_o]} \tag{16}$$

and $M(\xi',\eta';\xi,\eta)$ is a 2 x 2 matrix with the structure,

$$M(\xi',\eta';\ \xi,\eta) = M_o(\xi',\eta';\ \xi,\eta) + \varepsilon^2\, M_2(\xi',\eta';\ \xi,\eta)$$

where
$$M_o(\xi',\eta',\xi,\eta) = \begin{Vmatrix} \tau - \tau' - (\eta - \eta')\tau_\eta & (\eta - \eta')\tau_\xi \\ (\xi - \xi')\tau_\eta & \tau - \tau' - (\xi - \xi')\tau_\xi \end{Vmatrix}$$

$$M_2(\xi',\eta';\ \xi,\eta) = (\tau - \tau') \begin{Vmatrix} \tau_\xi^{\,2} & \tau_\xi\tau_\eta \\ \tau_\xi\tau_\eta & \tau_\eta^{\,2} \end{Vmatrix} = (\tau - \tau')\ N_2(\xi,\eta) \tag{17}$$

It may be noted that ε enters into eq. (15) in a more complicated way than the usual Fredholm equation of the second kind since ε also appears in a non-linear way in ρ_o. For this reason simple iteration of eq. (15) does not yield an expansion of $\psi(\xi,\eta)$ in powers of ε.

4. THE SCATTERED FIELD

Above the surface ($z > f(x,y)$) the total magnetic field is expressed in non-scaled coordinates by

$$\vec{H}(x') = \vec{H}^{ext}(x') - i\,\frac{1}{4\sqrt{2\pi}}\ k^3 \int d\sigma\ \ [\vec{n}(x) \times \vec{H}(x)] \times \vec{r}(x,x')\ \frac{H^{(1)}_{3/2}(kr)}{(kr)^{3/2}} \tag{18}$$

where x' now refers to an arbitrary point off the surface (coordinates: x', y', z') so

$$r = [(x - x')^2 + (y - y')^2 + (f - z')^2]^{\frac{1}{2}} \tag{19}$$

Eqs. (18) and (19) are easily expressed in scaled coordinates. From eq. (18) the scattered field is readily identified as the surface integral term with horizontal components given in two component column vector form by,

$$H_h^s(x') = - i \frac{H_o}{4\sqrt{2\pi}} \alpha^3 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \frac{H_{3/2}^{(1)}(\alpha\rho_o')}{(\alpha\rho_o')^{3/2}} M^s(x';\, \xi,\eta)\, \psi(\xi,\eta) \tag{20}$$

in which from eq. (19),

$$\rho_o' = [(\xi - \frac{kx'}{\alpha})^2 + (\eta - \frac{ky'}{\alpha})^2 + (\frac{kz'}{\alpha} - \epsilon\tau)^2]^{\frac{1}{2}} \tag{21}$$

and the 2 x 2 matrix $M^s(x';\xi,\eta)$ has the structure,

with

$$M_o^s(x';\, \xi,\eta) = - \frac{kz'}{\alpha}\; 1 = - \frac{kz'}{\alpha} \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}$$

$$M_1^s(x';\, \xi,\eta) = \begin{Vmatrix} \tau - (\eta - \frac{ky'}{\alpha})\tau_\eta & (\eta - \frac{ky'}{\alpha})\tau_\xi \\ (\xi - \frac{kx'}{\alpha})\tau_\eta & \tau - (\xi - \frac{kx'}{\alpha})\tau_\xi \end{Vmatrix}$$

$$M_2^s(x';\, \xi,\eta) = - \frac{kz'}{\alpha} N_2(\xi,\eta) \tag{22}$$

($N_2(\xi,\eta)$ is as defined implicitly in eq. (17)).

For the vertical component of the scattered field eq. (18) gives,

$$H_z^s(x') = i \frac{H_o}{4\sqrt{2\pi}} \alpha^3 \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta \frac{H_{3/2}^{(1)}(\alpha\rho_o')}{(\alpha\rho_o')^{3/2}} [V_1(x';\xi,\eta)\,\psi_1(\xi,\eta) + V_2(x';\xi,\eta)\,\psi_2(\xi,\eta)] \tag{23}$$

where

$$V_1(x';\xi,\eta) = \xi - \frac{kx'}{\alpha} + \varepsilon^2 [(\xi - \frac{kx'}{\alpha})\tau_\xi^2 + (\eta - \frac{ky'}{\alpha})\tau_\xi \tau_\eta]$$

$$V_2(x';\xi,\eta) = \eta - \frac{ky'}{\alpha} + \varepsilon^2 [(\xi - \frac{kx'}{\alpha})\tau_\xi \tau_\eta + (\eta - \frac{ky'}{\alpha})\tau_\xi{}^2] \tag{24}$$

The program for the calculation of the exact form of the scattered field is now formally complete, and proceeds by two stages. First, the surface currents are calculated from eq. (15). Then the scattered field components are calculated at an arbitrary field point by eqs. (20) and (23). Both steps present formidable computational difficulties, and, for the case of an arbitrary surface $\tau(\xi,\eta)$ it is necessary to resort to numerical analysis for usable results. Before turning to the necessary numerical procedures it is of some interest to see what information can be extracted from the basic theory.

5. SOLUTION FOR THE CASE $[\varepsilon << 1]$

As mentioned before simple iteration of eq. (15) does not produce a power series expansion of $\psi(\xi,\eta)$ in ε. However, such a series expansion can be obtained from eq. (15) with some restrictions on $\tau(\xi,\eta)$. The incident field is easily expanded in powers of ε,

$$\psi^{ext}(\xi) = e^{i\alpha\xi \cos\theta_o} [1 - i\alpha\varepsilon\tau(\xi) \sin\theta_o - \frac{1}{2}\alpha^2\varepsilon^2\tau^2(\xi) \sin^2\theta_o \ldots] \tag{25}$$

It is assumed that $\psi = \psi^{(0)} + \varepsilon\psi^{(1)} + \varepsilon^2\psi^{(2)} + \ldots$ The difficulty arises in the expansion of $H_{3/2}^{(1)}(\alpha\rho_o)/(\alpha\rho_o)^{3/2}$. If the surface satisfies the Lipshitz condition,

$$\varepsilon|\tau - \tau'| < [(\xi - \xi')^2 + (\eta - \eta')^2]^{\frac{1}{2}} \tag{26}$$

then $H_{3/2}^{(1)}(\alpha\rho_o)/(\alpha\rho_o)^{3/2}$ can be expanded using the Gegenbauer addition formula,

$$\frac{H_{3/2}^{(1)}(\alpha\rho_o)}{(\alpha\rho_o)^{3/2}} = \frac{H_{3/2}^{(1)}(\alpha\rho_1)}{(\alpha\rho_1)^{3/2}} + O(\varepsilon^2) \tag{27}$$

where $\rho_1 = [(\xi - \xi')^2 + (\eta - \eta')^2]^{\frac{1}{2}}$. If we restrict our efforts to a second order calculation then the presence of a factor of ε in eq. (15) before the integral renders the $O(\varepsilon^2)$ term effectively proportional to ε^3 and is hence discardable. Moreover, $M(\xi',\eta'; \xi,\eta)$ may be replaced by $M_o(\xi',\eta'; \xi,\eta)$ for the same reason. The resulting integral equation is considerably more tractable. Substitution of the assumed power series expansions in ε yield a set of equations giving $\psi^{(0)}$, $\psi^{(1)}$ in terms of $\psi^{(0)}$, etc. In order to carry out the integrations, it is necessary to introduce the Fourier expansion of the surface,

$$\tau(\xi,\eta) = \int_{-\infty}^{\infty} d\kappa \int_{-\infty}^{\infty} d\mu \; c(\kappa,\mu) \; e^{i(\kappa\xi + \mu\eta)}$$

Since $\tau(\xi,\eta)$ is real, $c(\kappa,\mu)^* = c(-\kappa,-\mu)$.

The integrations encountered are somewhat complicated and much use is made of the representation,

$$\frac{H_\nu^{(1)}(z)}{z^\nu} = \frac{1}{\pi i 2^\nu} \int_c du\, u^{-\nu-1}\, e^{u-z^2/4u} \tag{28}$$

c is a contour as shown in Fig. 1. Once the integration techniques are in hand the procedure to calculate $\psi^{(0)} + \epsilon\psi^{(1)} + \epsilon^2\psi^{(2)}$ is straightforward but tedious. Only the result need be quoted here and is,

$$\begin{aligned}
\psi(\xi,\eta) &= 2e^{i\alpha\xi\cos\theta_o}\,\psi_o^{ext} \\
&+ 2(i\alpha^3\epsilon)\int_\infty^\infty d\rho \int_\infty^\infty d\chi\; c(\alpha(\rho-\cos\theta_o),\alpha\chi)e^{i\alpha(\rho\xi+\chi\eta)}[N(\rho,\chi,\cos\theta_o,0) \\
&\qquad - 1\sin\theta_o]\psi_o^{ext} \\
&+ 2(i\alpha^3\epsilon)^2 \int_\infty^\infty d\rho \int_\infty^\infty d\chi \int_\infty^\infty d\rho' \int_\infty^\infty d\chi'\; c(\alpha(\rho-\rho'),\alpha(\chi-\chi'))\,c(\alpha(\rho'-\cos\theta_o),\alpha\chi') \\
&\qquad e^{i\alpha(\rho\xi+\chi\eta)} \\
&\times \{N(\rho,\chi;\rho',\chi')\,[N(\rho',\chi';\cos\theta_o,0) - 1\sin\theta_o] + \tfrac{1}{2}\,1\sin^2\theta_o\}\,\psi_o^{ext}
\end{aligned} \tag{29}$$

where

$$N(\rho,\chi;\rho',\chi') = \frac{1}{(1-\rho^2-\chi^2)^{\frac{1}{2}}} \begin{Vmatrix} \rho^2+\chi\chi'-1 & (\rho-\rho')\chi \\ \rho(\chi-\chi') & \rho\rho'+\chi^2-1 \end{Vmatrix} + 1(1-\rho'^2-\chi'^2)^{\frac{1}{2}} \tag{30}$$

In both eq. (29) and eq. (30), 1 stands for the unit 2 x 2 matrix. The zeroth order term $\psi^{(0)} = 2e^{i\alpha\xi \cos\theta_o} \psi_o^{ext}$ is recognized as the flat surface current and substitution of this term into eqs. (20) and (23) gives a reflected wave exhibiting the oldest known fact about reflection, angle of incidence equals angle of reflection.

From eqs. (20) and (23) it is seen that the ε expansion of the fields involves the ε expansion of $H_{3/2}^{(1)}(\alpha\rho_o')/(\alpha\rho_o')^{3/2}$. This can be made with the aid of the Gegenbauer addition formula again as follows. Introducing,

$$\rho_1' = \left[(\xi - \frac{kx'}{\alpha})^2 + (\eta - \frac{ky'}{\alpha})^2 + (\frac{kz'}{\alpha})^2\right]^{\frac{1}{2}} \tag{31}$$

eq. (21) can be written

$$\rho_o' = \left[\rho_1'^2 - 2\rho_1' \varepsilon\tau (\frac{kz'}{\alpha\rho_1'}) + (\varepsilon\tau)^2\right]^{\frac{1}{2}} \tag{32}$$

The requirement $\frac{kz'}{\alpha} > \varepsilon\tau_{max}$ (i.e., $z' > f_{max}$) allows an expansion of $H_{3/2}^{(1)}(\alpha\rho_o')/(\alpha\rho_o')^{3/2}$ in powers of ε, again using the Gegenbauer addition theorem. This expansion is carried to second order and the result inserted in eqs. (20) and (23). Moreover, as eq. (22) shows, $M_o^s(x'; \xi,\eta)$ is a power series in ε to second order, as are V_1 and V_2 (eq. (24)). Using the power series expansion in ε for $\psi(\xi,\eta)$ results in a lengthy calculation which determines the scattered field as a power series in ε. Since the expressions which result are somewhat cumbersome they are included as Appendix I. The points to note concerning this expression for the scattered fields are (1) the result is valid for the Lipshitz condition of eq. (26), and (2) the result is valid only in the region $z' > f_{max}$. Of course, it would be expected that the solution failed if ε^2 were of significant size.

Several other features of these ε power series solutions are noteworthy. First, with the assumption of periodicity in the surface so that

$$\tau(\xi,\eta) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{mn} \, e^{i\frac{2\pi}{L}(m\xi + n\eta)} \tag{33}$$

or, equivalently,

$$c(\kappa,\mu) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_{mn} \, \delta(\kappa - \frac{2\pi m}{L}) \, \delta(\mu - \frac{2\pi n}{L}) \tag{34}$$

the expansions in ε simply reproduce the physical optics theory of Rice. It is, of course, not clear that this equivalence persists to higher order than two since for cubic, or higher, terms the power series expansions involve corrections to the Green's function itself. A second result is implicit in the ε expansion of the Green's function $H^{(1)}_{3/2}(\alpha\rho_o')/(\alpha\rho_o')^{3/2}$. The result is a series of Green's functions dependent on the argument ρ_1'. It can be seen from the structure of ρ_1' (eq. (31)) that the series is interpretable as several partial Green's functions assembling the field from a current distribution given on $z' = 0 = f_{min}$. This is true in spite of the restriction $z' > f_{max}$ for the validity of the ε expansion of the fields.

When the condition $0 < z' < f_{max}$ applies, as for example when the field point is located in a "valley" of the surface, then the Gegenbauer expansion of $H^{(1)}_{3/2}(\alpha\rho_o')/(\alpha\rho_o)^{3/2}$ requires modification. Consider ρ_1' as a function of ξ and η with fixed x', y', z'. A cross-section of this function, together with the surface is shown in Fig. 2. The minimum of $\rho_1'(\xi,\eta)$ occurs at $\xi = kx'/\alpha$, $\eta = ky'/\alpha$ and is equal to kz'/α. For large ξ and η, $\rho_1'(\xi,\eta)$ is asymptotic to a cone of semi-vertex angle $\pi/4$ with axis $\xi = kx'/\alpha$, $\eta = ky'/\alpha$ and vertex located at $z' = 0$. In general the surface defined by $\rho_1'(\xi,\eta)$ intersects $\tau(\xi,\eta)$ in a finite number of regions

near $\xi = kx'/\alpha$, $\eta = ky'/\alpha$. For those portions of the surface where $\rho_1' < \epsilon\tau$ there corresponds a set, denoted by $\bar{\gamma}$, of disjoint regions in the ξ,η plane. For the regions in the ξ,η plane outside $\bar{\gamma}$, which is denoted γ, the inequality $\rho_1' > \epsilon\tau$ is still valid. In making the ϵ expansion of $H_{3/2}^{(1)}(\alpha\rho_o')/(\alpha\rho_o')^{3/2}$ for region γ the Gegenbauer formula is used exactly as before. For the region a new length is introduced:

$$\rho_2' = [(\xi - \frac{kx'}{\alpha})^2 + (\eta - \frac{ky'}{\alpha})^2 + (\epsilon\tau_{max} - \frac{kz'}{\alpha})^2]^{\frac{1}{2}} \tag{35}$$

so that ρ_o' may be written,

$$\rho_o' = [(\rho_2')^2 + 2\rho_2' \epsilon(\tau_{max} - \tau) \frac{1}{\rho'_2} (\frac{kz'}{\alpha} - \epsilon\tau_{max}) + \epsilon^2(\tau_{max} - \tau)^2]^{\frac{1}{2}} \tag{36}$$

Since $\epsilon(\tau_{max} - \tau) \leqslant \epsilon\tau_{max} - kz'/\alpha$ the Gegenbauer addition formula may now be used to generate a series of partial Green's functions whose argument is $\alpha\rho_2'$. From the structure of ρ_2' it is seen that the partial Green's functions are, in the region $\bar{\gamma}$ propagating waves ("assembling the field") from current distributions located on $z' = f_{max}$ (i.e., $kz'/\alpha = \epsilon\tau_{max}$). These currents are <u>above</u> the field point and as seen at field point constitute <u>downward</u> propagating waves. The rather mathematical requirements of the Gegenbauer addition formula are thus seen to produce the missing downward propagating waves.

6. RESULTS FOR A PERIODIC SURFACE

In this section an analysis of the case where $\tau(\xi,\eta) = \tau(\xi) = \tau(\xi + 2)$ is undertaken to provide further comparison with the physical optics solution. The imposition of one-dimensional variation of the surface is made solely in the

interest of shortening the analysis. The techniques to be presented work as well for the full case $\tau = \tau(\xi,\eta)$ as the "one-dimensional" surface $\tau = \tau(\xi)$. The choice of period equal to two is made largely in the interest of facilitating the numerical solution of the source integral equation (eq. (15)). Again the techniques presented are not dependent on this choice of periodicity; only numerical quadrature formulae need be altered.

The first task is reduction of eq. (15) in consequence of $\tau = \tau(\xi)$. $M(\xi',\eta';\ \xi,\eta)$ is found to simplify enough to eliminate ψ_1 in the equation for ψ_2 so that only ψ_2 is involved. Since $\psi_2(\xi,\ \eta+\Delta) - \psi_2(\xi,\eta)$ satisfies a homogeneous integral equation (Fredholm integral equation of the first kind) for arbitrary Δ it can be assumed that $\psi_2(\xi,\eta) = \psi_2(\xi)$. This proves to be a sufficient condition to eliminate ψ_2 in the equation for ψ_1. The two equations thus become fully decoupled when $\tau = \tau(\xi)$. All that is necessary for reduction to two Fredholm equations of the second kind for the two functions $\psi_1(\xi)$ and $\psi_2(\xi)$ is the η integral of $H^{(1)}_{3/2}(\alpha\rho_o)/(\alpha\rho_o)^{3/2}$. This is easily accomplished using the contour integral representation of eq. (28). There results,

$$\psi_1(\xi') = 2ae^{i\alpha[\xi' \cos\theta_o - \epsilon\tau' \sin\theta_o]} - i\frac{\epsilon\alpha}{2}\int_{-\infty}^{\infty} d\xi\ \frac{H_1^{(1)}(\alpha\rho_1)}{\rho_1}(\tau-\tau')(1+\epsilon^2\tau_\xi^2)\ \psi_1(\xi)$$

$$\psi_2(\xi') = 2be^{i\alpha[\xi' \cos\theta_o - \epsilon\tau' \sin\theta_o]} - i\frac{\epsilon\alpha}{2}\int_{-\infty}^{\infty} d\xi\ \frac{H_1^{(1)}(\alpha\rho_1)}{\rho_1}[\tau-\tau' - (\xi-\xi')\tau_\xi]\ \psi_2(\xi) \tag{37}$$

a and b again depend on incident polarization, and with $\tau' = \tau(\xi')$,

$$\rho_1 = [(\xi - \xi')^2 + \epsilon^2(\tau - \tau')^2]^{\frac{1}{2}} \tag{38}$$

For the special case of $\tau(\xi) = \cos\pi\xi$ these equations have been studied by Zaki

and Neureuther.[6]

Either of the eqs. (37) can be used to illustrate the method of solution. The TM mode is selected here and accordingly b = -1 so the relevant equation is

$$\psi(\xi') = -2e^{i\alpha[\xi' \cos\theta_o - \epsilon\tau' \sin\theta_o]} - i\frac{\epsilon\alpha}{2}\int_{\infty}^{\infty} d\xi \, \frac{H_1^{(1)}(\alpha\rho_1)}{\rho_1}\left[\tau-\tau' - (\xi-\xi')\tau_\xi\right]\psi(\xi) \tag{39}$$

where the 2 subscript has been dropped. For this case, since a = 0, $\psi_1(\xi) = 0$. Since $\psi(\xi + 2) = \psi(\xi)$ it follows that $\psi(\xi + 2n) = \psi(\xi)$ for n = 0, ±1, ±2, ... Eq. (39) is simply a Fredholm equation of the second kind with kernel,

$$K(\xi',\xi) = -i\,\frac{\epsilon\alpha}{2}\,\frac{H_1^{(1)}(\alpha\rho_1)}{\rho_1}\left[\tau - \tau' - (\xi - \xi')\,\tau_\xi\right] \tag{40}$$

Because of the periodicity imposed on $\tau(\xi)$ the kernel is seen to have the periodicity,

$$K(\xi' + 2n,\ \xi + 2n) = K(\xi',\ \xi) \tag{41}$$

As it stands the inhomogeneous term (in the sense of a Fredholm equation) has no simple symmetry. However, if $f(\xi) = -2e^{i\alpha[\xi \cos\theta_o - \epsilon\tau \sin\theta_o]}$, then,

$$f(\xi + 2) = e^{i2\alpha \cos\theta_o}\ f(\xi) \tag{42}$$

so the restriction of θ_o to satisfy $2\alpha \cos\theta_o = 2\pi p$, p = 0, 1, 2, ... with $\pi p < \alpha$ yields the same basic symmetry as the surface. Eq. (39) is now written

$$\psi(\xi') = f(\xi') + \int_{\infty}^{\infty} d\xi \quad K(\xi',\xi)\ \psi(\xi) \tag{43}$$

From which it is easy to show that $\psi(\xi + 2n) - \psi(\xi)$ satisfies a homogeneous integral equation and is thus equal to zero. If ξ' is confined to the range $-1 < \xi' < 1$ and the integral written

$$\int_{-\infty}^{\infty} d\xi\ K(\xi',\xi)\ \psi(\xi) = \sum_{n=-\infty}^{\infty} \int_{2n-1}^{2n+1} d\xi\ K(\xi',\xi)\ \psi(\xi)$$

$$= \int_{-1}^{1} d\xi\ \left[\sum_{n=-\infty}^{\infty} K(\xi',\xi + 2n)\right] \psi(\xi) \tag{44}$$

then eq. (43) is reduced to a form that is suitable for numerical solution.

$$\psi(\xi') = f(\xi') + \int_{-1}^{1} d\xi\ K_f(\xi',\xi)\ \psi(\xi) \tag{45}$$

where the folded kernel, $K_f(\xi',\xi)$ has been introduced.

$$K_f(\xi',\xi) = \sum_{n=-\infty}^{\infty} K(\xi',\ \xi + 2n) \tag{46}$$

From this form it follows at once that $K_f(\xi' + 2n,\ \xi) = K_f(\xi',\ \xi = 2n) = K_f(\xi',\xi)$. It is, of course, possible to determine $K_f(\xi',\xi)$ directly from eq. (44) but the structure of $K(\xi',\ \xi + 2n)$ makes the convergence of the series very slow indeed. The reason for this may be seen from eq. (40). As ρ_1 increases so does the term in square brackets in eq. (40). Therefore, the decrease in successive terms depends on the decrease in $H_1^{(1)}\ (\alpha\rho_1)$ which is proportional to $\rho_1^{-\frac{1}{2}}$, a slowly decreasing function.

It proves possible to improve the situation by using the integral representation, eq. (28), together with the Poisson Sum Formula.[7] From,

$$K_f(\xi',\xi) = K(\xi',\xi) + \sum_{n=1}^{\infty} [K(\xi', \xi + 2n) + K(\xi', \xi - 2n)] \tag{47}$$

and eq. (28) there results,

$$\sum_{n=1}^{\infty} [K(\xi', \xi + 2n) + K(\xi', \xi - 2n)] =$$

$$i\varepsilon\alpha^2 [(\xi-\xi')\tau_\xi - (\tau-\tau')] \frac{1}{2\pi i} \int_C dv\, v^{-2} e^{v-\frac{\alpha^2}{4v}\rho_1^2} \sum_{n=1}^{\infty} e^{-\frac{n^2\alpha^2}{v}} \cosh \frac{n\alpha^2}{v} (\xi-\xi')$$

$$-2i\varepsilon\alpha^2\tau_\xi \frac{1}{2\pi i} \int_C dv\, v^{-2} e^{v-\frac{\alpha^2}{4v}\rho_1^2} \sum_{n=1}^{\infty} n\, e^{-\frac{n^2\alpha^2}{v}} \sinh \frac{n\alpha^2}{v} (\xi-\xi') \tag{48}$$

Applying the Poisson Sum Formula to the first sum on the right of this equation gives:

$$\sum_{n=1}^{\infty} e^{-\frac{n^2\alpha^2}{v}} \cosh \frac{n\alpha^2}{v} (\xi-\xi') = -\tfrac{1}{2} + \frac{\sqrt{\pi}}{2\alpha} v^{\frac{1}{2}} e^{\frac{\alpha^2}{4v}(\xi-\xi')^2} \sum_{n=-\infty}^{\infty} e^{i\pi n(\xi-\xi') - \frac{\pi^2 n^2}{\alpha^2} v} \tag{49}$$

The right hand side of this substituted in the first integral on the right of eq. (48) coupled with a change of variable in the integral $v(1 - \pi n^2/\alpha^2) \to v$ gives, after use of some basic definitions of the half-integral Hankel function, the contribution,

$$-K(\xi',\xi) + \tfrac{1}{2}[(\xi-\xi')\tau_\xi - (\tau-\tau')] \frac{1}{|\tau-\tau'|} \sum_{n=-\infty}^{\infty} e^{i\pi n(\xi-\xi') + i\alpha\varepsilon|\tau-\tau'|(1-\frac{n^2\pi^2}{\alpha^2})^{\frac{1}{2}}} \tag{50}$$

The second integral is treated the same way and yields

$$-\tfrac{1}{2}\,\frac{(\xi-\xi')\tau_\xi}{|\tau-\tau'|}\sum_{n=-\infty}^{\infty} e^{i\pi n(\xi-\xi') + i\alpha\varepsilon(1-n^2\pi^2/\alpha^2)^{\frac{1}{2}}|\tau-\tau'|}$$

$$+\tfrac{1}{2}\,\varepsilon\tau_\xi\sum_{n=-\infty}^{\infty}\frac{n\pi/\alpha}{(1-n^2\pi^2/\alpha^2)^{\frac{1}{2}}}\, e^{i\pi n(\xi-\xi') + i\alpha\varepsilon(1-n^2\pi^2/\alpha^2)^{\frac{1}{2}}|\tau-\tau'|} \tag{51}$$

so that $K_f(\xi',\xi)$ is given by

$$K_f(\xi',\xi) = -\tfrac{1}{2}\,\frac{\tau-\tau'}{|\tau-\tau'|}\sum_{n=-\infty}^{\infty} e^{i\pi n(\xi-\xi') + i\alpha\varepsilon(1-n^2\pi^2/\alpha^2)^{\frac{1}{2}}|\tau-\tau'|}$$

$$+\tfrac{1}{2}\,\varepsilon\tau_\xi\sum_{n=-\infty}^{\infty}\frac{\pi n/\alpha}{(1-n^2\pi^2/\alpha^2)^{\frac{1}{2}}}\, e^{i\pi n(\xi-\xi') + i\alpha\varepsilon(1-n^2\pi^2/\alpha^2)^{\frac{1}{2}}|\tau-\tau'|} \tag{52}$$

This form exhibits the periodicity of $K_f(\xi',\xi)$ explicitly and offers a considerable improvement in calculational time. It should be noted that $K_f(\xi',\xi')$ is not defined by the above equation, but a simple argument based on the fundamental definition of $K_f(\xi',\xi)$ gives $K_f(\xi',\xi') = K(\xi',\xi') = (\varepsilon/2\pi)(\tau'_{\xi\xi}/1 + \varepsilon^2\tau'^2_{\xi})$.

7. CONCLUSION

An exact solution for scattering from a rough surface has been presented which has as its single restriction that the surface is perfectly conducting. Calculation of the surface currents forms the first step while calculation of the scattered field from these as sources constitutes the second step. The problem of downward propagating waves near the surface has been shown to be a natural consequence of step two. The general problem was then limited to a surface

exhibiting periodicity, $f(x,y) = f(x) = f(x + 2\Lambda)$ and improved methods for calculating the appropriate kernels of the source integral equations were presented. The results of the theory presented were shown to be in accord with the small roughness ($\epsilon << 1$) theory of Rice.

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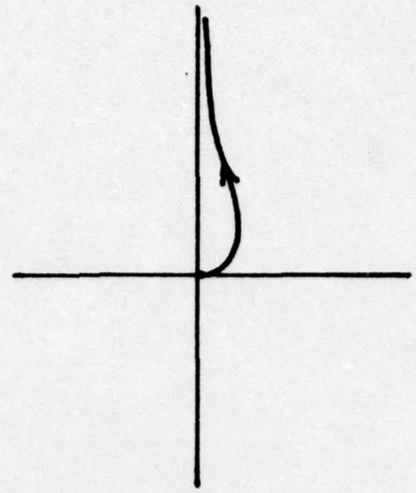

Fig. 1

Fig. 1. Contour for the integral representation of $H_{\nu}^{(1)}(z)/z^{\nu}$.

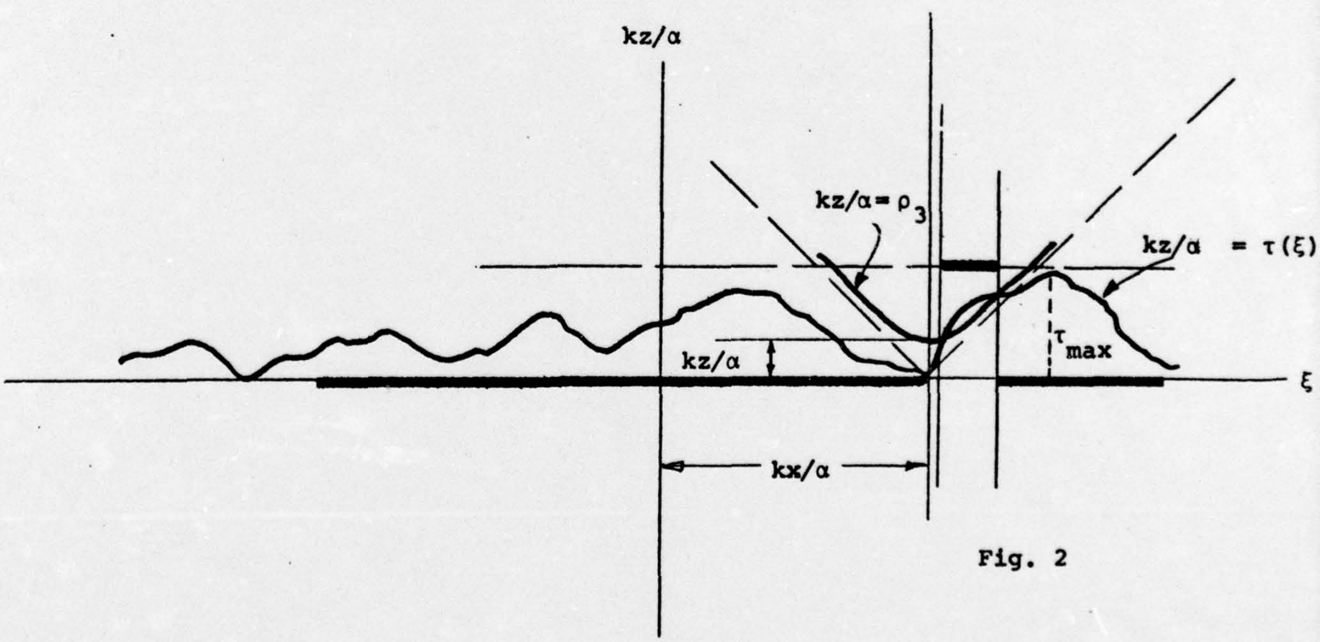

Fig. 2

Fig. 2. For the specific surface shown, the equivalent flat surface current distribution is shown as a heavy line. The portion located on $kz/\alpha = \tau_{max}$ generates downward as well as upward propagating waves.

APPENDIX I

Equations (20) through (24) give the scattered magnetic field in terms of surface current. From eq. (22) the ε power series expansion of $M^s(x',\xi,\eta)$ is obtained, as is the ε power series expansion of $V_1(x',\xi,\eta)$ and $V_2(x',\xi,\eta)$ from eq. (24). The ε power series for $\psi(\xi,\eta)$ has been given in eq. (29). The remaining problem is the ε series for $H^{(1)}_{3/2}(\alpha\rho_o')/(\alpha\rho_o')^{3/2}$ with ρ_o' as in eq. (21). For comparison with the results of Rice the assumption $z' > \tau_{max}$ is made and the Gegenbauer addition theorem gives,

$$\frac{H^{(1)}_{3/2}(\alpha\rho_o')}{(\alpha\rho_o')^{3/2}} = 2^{3/2}\,\Gamma(3/2)\sum_{\ell=0}^{\infty}(3/2+\ell)\,\frac{H^{(1)}_{\ell+3/2}(\alpha\rho_1')}{(\alpha\rho_1')^{3/2}}\;\frac{J_{\ell+3/2}(\varepsilon\alpha\tau)}{(\varepsilon\alpha\tau)^{\ell+3/2}}\;C_\ell^{(3/2)}\left(\frac{kz'}{\alpha\rho_1'}\right) \tag{I-1}$$

where ρ_1' is given by eq. (31), and $C_\ell^{(3/2)}(x)$ are the usual ultraspherical polynomials: $C_o^{(3/2)}(x) = 1$, $C_1^{(3/2)}(x) = 3x$, $C_2^{(3/2)}(x) = 3/2(5x^2-1)$ ... *etc.* The ε expansion is made by expanding $J_{\ell+3/2}(\varepsilon\alpha\tau)/(\varepsilon\alpha\tau)^{\ell+3/2}$ and retaining terms to order ε^2. We set,

$$\frac{H^{(1)}_{3/2}(\alpha\rho_o')}{(\alpha\rho_o')^{3/2}} = K_o + \varepsilon K_1 + \varepsilon^2 K_2 \tag{I-2}$$

Using the expansions for $\psi(\xi,\eta)$ and $M^s(x',\xi,\eta)$ the integrand of eq. (20) or (23) can be written as a power series in ε where each integral is done using the result derived from eq. (28)

$$I_\nu(\rho,\chi;\beta) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\; \frac{H^{(1)}_\nu[(\xi^2+\eta^2+\beta^2)^{1/2}]}{(\xi^2+\eta^2+\beta^2)^{\nu/2}}\; e^{i(\rho\xi+\chi\eta)}$$

$$= 2\pi/\beta^{\nu-1} \; (1 - \rho^2 - \chi^2)^{\nu/2-1/2} \; H_{\nu-1}^{(1)} \left[\beta(1 - \rho^2 - \chi^2)^{\frac{1}{2}}\right] \tag{I-3}$$

To order ε^2 ten integrals are involved for $H_h^s\,(x')$ while seven suffice for $H_z^s\,(x')$. As the calculations are elementary though very tedious only the results are given.

$$E(x,\ y,\ z;\ \rho,\ \chi) = e^{ik[x\rho + y\chi + z(1 - \rho^2 - \chi^2)^{\frac{1}{2}}]}$$

$$H_{h0}^s\,(x) = H_o \psi_o^{ext}\; e^{ik(x \cos\theta_o + z \sin\theta_o)}$$

$$H_{h1}^s\,(x) = -i\varepsilon 2H_o\alpha^3 \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\chi \;\; c(\alpha(\rho - \cos\theta_o),\ \alpha\chi)\ E(x,\ y,\ z;\ \rho,\ \chi)$$

$$\times \frac{1}{(1 - \rho^2 - \chi^2)^{\frac{1}{2}}} \begin{Vmatrix} 1 - \rho^2 - \chi^2 + \chi^2 & -(\rho - \cos\theta_o)\chi \\ -\rho\chi & 1 - \rho^2 - \chi^2 + (\rho - \cos\theta_o)\rho \end{Vmatrix} \psi_o^{ext}$$

$$H_{h2}^s\,(x) = -\varepsilon^2 2H_o\alpha^6 \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\chi \int_{-\infty}^{\infty} d\rho' \int_{-\infty}^{\infty} d\chi'\ c(\alpha(\rho-\rho'),\ \alpha(\chi-\chi'))$$

$$c(\alpha(\rho'-\cos\theta_o),\ \alpha\chi')\ E(x,\ y,\ z;\ \rho,\ \chi)$$

$$\times \frac{1}{(1 - \rho^2 - \chi^2)^{\frac{1}{2}}(1 - \rho'^2 - \chi'^2)^{\frac{1}{2}}} \begin{Vmatrix} 1 - \rho^2 - \chi^2 + \chi(\chi - \chi') & -(\rho - \rho')\chi \\ -\rho(\chi - \chi') & 1 - \rho^2 - \chi^2 + \rho(\rho - \rho') \end{Vmatrix}$$

$$\begin{Vmatrix} 1 - \rho'^2 - \chi'^2 + \chi'^2 & -(\rho' - \cos\theta_o)\chi' \\ -\rho'\chi' & 1 - \rho'^2 - \chi'^2 + \rho'(\rho' - \cos\theta_o) \end{Vmatrix} \psi_o^{ext}$$

$$H^{s}_{z0}(x) = -H_o \cos\theta_o\, e^{ik[x\cos\theta_o + z\sin\theta_o]}$$

$$H^{s}_{z1}(x) = -i\varepsilon 2H_o\alpha^3 \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\chi \;\; c(\alpha(\rho - \cos\theta_o), \alpha\chi)$$

$$E(x, y, z; \rho, \chi) \times (\chi\Gamma_o - \rho\sin\theta_o)$$

$$H^{s}_{z2}(x) = -\varepsilon^2 2H_o\alpha^6 \int_{-\infty}^{\infty} d\rho \int_{-\infty}^{\infty} d\chi \int_{-\infty}^{\infty} d\rho' \int_{-\infty}^{\infty} d\chi' \; c(\alpha(\rho-\rho'), \alpha(\chi-\chi'))$$

$$c(\alpha(\rho'-\cos\theta_o), \alpha\chi')\; E(x, y, z; \rho, \chi)$$

$$\times \frac{1}{(1-\rho'^2-\chi'^2)^{\frac{1}{2}}} \{\rho[(1-\rho'^2)\sin\theta_o + (\rho'-\cos\theta_o)\chi'\Gamma_o]$$

$$-\chi[\rho'\chi'\sin\theta_o + (1-\chi'^2-\rho'\cos\theta_o)\Gamma_o]\}$$